Production Scheduling Model

DETERMINISTIC PERIODIC-REVIEW MODEL

Previously the basic EOQ model and some of its variations were dependent upon the assumption of a constant demand rate. When this assumption is relaxed, i.e., when the amounts that need to be withdrawn from inventory are allowed to vary from period to period, the *EOQ formula* no longer ensures a minimum-cost solution.

Consider the following periodic-review model. Planning is to be done for the next *n* periods regarding how much (if any) to produce or order to replenish inventory at the beginning of each of the periods. (The order to replenish inventory can involve either *purchasing* the units or *producing* them, but the latter case is far more common with applications of this model, so we mainly will use the terminology of *producing* the units.) The demands for the respective periods are *known* (but *not* the same in every period) and are denoted by

r_i = demand in period i, for i _ 1, 2, . . . , n.

These demands must be met on time. There is no stock on hand initially, but there is still time for a delivery at the beginning of period 1.

The costs included in this model are similar to those for the basic EOQ model:

K= setup cost for producing or purchasing any units to replenish inventory at beginning of period,

c =unit cost for producing or purchasing each unit,

h=holding cost for each unit left in inventory at end of period.

Note that this holding cost h is assessed only on inventory left at the end of a period. There also are holding costs for units that are in inventory for a portion of the period before being withdrawn to satisfy demand. However, these are *fixed* costs that are independent of the inventory policy and so are not relevant to the analysis. Only the *variable* costs that are affected by which inventory policy is chosen, such as the extra holding costs that are incurred by carrying inventory over from one period to the next, are relevant for selecting the inventory policy.

By the same reasoning, the unit cost c is an irrelevant fixed cost because, over all the time periods, all inventory policies produce the same number of units at the same cost. Therefore, c will be dropped from the analysis hereafter.

The objective is to minimize the total cost over the n periods. This is accomplished by ignoring the fixed costs and minimizing the total variable cost over the n periods, as illustrated by the following example.

Example. An airplane manufacturer specializes in producing small airplanes. It has just received an order from a major corporation for 10 customized executive jet airplanes for the use of the corporation's upper management. The order calls for three of the airplanes to be delivered (and paid for) during the upcoming winter months (period 1), two more to be delivered during the spring (period 2), three more during the summer (period 3), and the final two during the fall (period 4).

Setting up the production facilities to meet the corporation's specifications for these airplanes requires a setup cost of \$2 million. The manufacturer has the capacity to produce all 10 airplanes within a couple of months, when the winter season will be under way. However, this would necessitate holding seven of the airplanes in inventory, at a cost of $200,000 per airplane per period, until their scheduled delivery times. To reduce or eliminate these substantial holding costs, it may be worthwhile to produce a smaller number of these airplanes now and then to repeat the setup (again incurring the cost of $2 million) in some or all of the subsequent periods to produce additional small numbers. Management would like to determine the least costly production schedule for filling this order.

Thus, using the notation of the model, the demands for this particular airplane during the four upcoming periods (seasons) are

$r1 = 3$, $r2 = 2$, $r3 = 3$, $r4 = 2$.

Using units of millions of dollars, the relevant costs are

$K = 2$, $h = 0.2$.

The problem is to determine how many airplanes to produce (if any) during the beginning of each of the four periods in order to minimize the total variable cost.

The high setup cost K gives a strong incentive not to produce airplanes every period and preferably just once. However, the significant holding cost h makes it undesirable to carry a large inventory by producing the entire demand for all four periods (10 airplanes) at the beginning. Perhaps the best approach would be an intermediate strategy where airplanes are produced more than once but less than four times. For example, one such feasible solution (but not an optimal one) is depicted in Fig. 1, which shows the evolution of the inventory level over the next year that results from producing three airplanes at the beginning of the first period, six airplanes at the beginning of the second period, and one airplane at the beginning of the fourth period. The dots give the inventory levels after any production at the beginning of the four periods.

How can the optimal production schedule be found? For this model in general, production (or purchasing) is automatic in period 1, but a decision on whether to produce must be made for each of the other n - 1 periods. Therefore, one approach to solving this model is to enumerate, for each of the 2^{n-1} combinations of production decisions, the possible quantities that can be produced in each period where production is to occur. This approach is rather cumbersome, even for moderate-sized n, so a more efficient method is desirable. Such a method is described next in general terms, and then we will return to finding the optimal production schedule for the example. Although the general method can be used when either producing or purchasing to replenish inventory, we now will only use the terminology of producing for definiteness.

An Algorithm

The key to developing an efficient algorithm for finding an *optimal inventory policy* (or equivalently, an *optimal production schedule*) for the above model is the following insight into the nature of an optimal policy.

An optimal policy (production schedule) produces *only* when the inventory level is *zero*. To illustrate why this result is true, consider the policy shown in Fig. 1 for the example. (Call it policy *A*.)

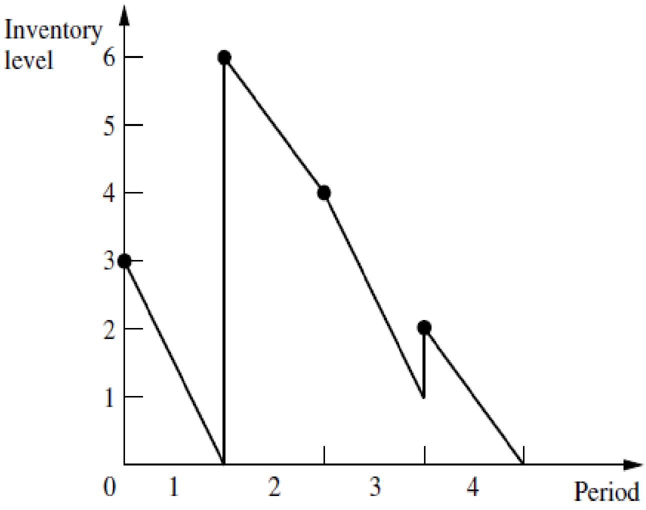

Fig 1. The inventory level

Policy *A* violates the above characterization of an optimal policy because production occurs at the beginning of period 4 when the inventory level is *greater than zero* (namely, one airplane). However, this policy can easily be adjusted to satisfy the above characterization by simply producing one less airplane in period 2 and one more airplane in period 4. This adjusted policy (call it *B*) is shown by the dashed line in Fig. 2 wherever *B* differs from *A* (the solid line). Now note that policy *B* *must* have less total cost than policy *A*. The setup costs (and the production costs) for both policies are the same. However, the holding cost is smaller for *B* than for *A* because *B* has less inventory than *A* in periods 2 and 3 (and the same inventory in the other periods). Therefore, *B* is better than *A*, so *A* cannot be optimal.

This characterization of optimal policies can be used to identify policies that are not optimal. In addition, because it implies that the only choices for the amount produced at the beginning of the i^{th} period are 0, r_i, $r_i + r_{i+1}$, . . . , or $r_i+r_{i+1}+....+ r_n$, it can be exploited to obtain an efficient algorithm that is related to the *deterministic dynamic programming* approach.

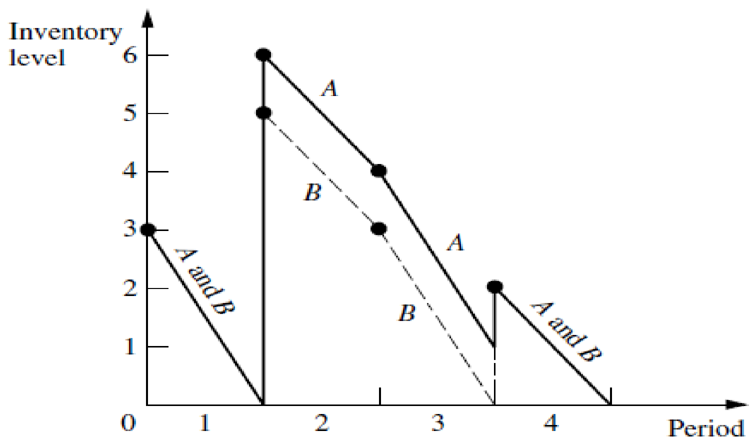

Fig.2 Comparison of two inventory policies A and B

In particular, define

$\boldsymbol{C_i}$= total variable cost of an optimal policy for periods i, i+ 1, . . . , n when period i starts with zero inventory (before producing), for i= 1, 2, . . . , n.

By using the dynamic programming approach of solving *backward* period by period, these C_i values can be found by first finding C_n, then finding C_{n+1}, and so on. Thus, after C_n, Cn-1, . . . , C_{i-1} are found, then Ci can be found from the *recursive relationship*

C_i = minimum $_{j=i,\, i+1,\, \ldots,\, n}\{C_{j+1} + K + h[r_{i+1} + 2r_{i+2} + 3r_{i+3} + \ldots + (j-i)rj]\}$,

where j can be viewed as an index that denotes the (end of the) period when the inventory reaches a zero level for the first time after production at the beginning of period i. In the time interval from period i through period j, the term with coefficient h represents the total *holding cost* over this interval. When j =n, the term C_{n+1} =0. The *minimizing value* of j indicates that if the inventory level does indeed drop to zero upon entering period i, then the production in period i should cover all demand from period i through this period j.

The algorithm for solving the model consists basically of solving for C_n, C_{n-1}, . . . , C_1 in turn. For i = 1, the minimizing value of j then indicates that the production in period 1 should cover the demand through period j, so the second production will be in period j +1. For i = j +1, the new minimizing value of j identifies the time interval covered by the second production, and so forth to the end. We will illustrate this approach with the example.

Application of the Algorithm to the Example

Returning to the airplane example, first we consider the case of finding C_4, the cost of the optimal policy from the beginning of period 4 to the end of the planning horizon:

C_4= C_5+2 =0+2=2.

To find C_3, we must consider two cases, namely, the first time after period 3 when the inventory reaches a zero level occurs at (1) the end of the third period or (2) the end of the fourth period. In the recursive relationship for C_3, these two cases correspond to (1) j = 3 and (2) j = 4. Denote the corresponding costs (the right-hand side of the recursive relationship with this j) by $C_3^{(3)}$ and $C_3^{(4)}$ respectively. The policy associated with $C_3^{(3)}$ calls for producing only for period 3 and then following the optimal policy for period 4, whereas the policy associated with $C_3^{(4)}$ calls for producing for periods 3 and 4. The cost C_3 is then the minimum of $C_3^{(3)}$ and $C_3^{(4)}$.

$C_3^{(3)}$ =C_4+2=2+2=4

$C_3^{(4)}$=C_5+2+0.2(2)=0+2+0.4=2.4

C_3 = min{4, 2.4} =2.4.

Therefore, if the inventory level drops to zero upon entering period 3 (so production should occur then), the production in period 3 should cover the demand for both periods 3 and 4. To find C_2, we must consider three cases, namely, the first time after period 2 when the inventory reaches a zero level occurs at (1) the end of the second period, (2) the end of the third period, or (3) the end of the fourth period. In the recursive relationship for C_2, these cases correspond to (1) j =2, (2) j = 3, and (3) j = 4, where the corresponding costs are

$C_2^{(2)}$= C_3+2=2.4+2=4.4

$C_2^{(3)}$=C_4+2+0.2(3)=2+2+0.6=4.6

$C_2^{(4)}$= C_5+2+ 0.2[3+2(2)] = 0 +2 + 1.4 =3.4.

C_2= min {4.4, 4.6, 3.4} =3.4.

this production should cover the demand for all the remaining periods.

Finally, to find C_1, we must consider four cases, namely, the first time after period 1when the inventory reaches zero occurs at the end of (1) the first period, (2) the second period, (3) the third period, or (4) the fourth period. These cases correspond to j=1, 2, 3, 4 and to the costs $C_1^{(1)}$, $C_1^{(2)}$, $C_1^{(3)}$, $C_1^{(4)}$, respectively. The cost C_1 is then the minimum of $C_1^{(1)}$, $C_1^{(2)}$, $C_1^{(3)}$, $C_1^{(4)}$.

$C_1^{(1)} = C2 +2 = 3.4 +2 = 5.4.$

$C_1^{(2)} = C3 +2 + 0.2(2) = 2.4 +2 +0.4 =4.8.$

$C_1^{(3)} =C4 +2 +0.2[2 +2(3)] =2 +2 + 1.6 =5.6.$

$C_1^{(4)} =C5 +2 +0.2[2 +2(3) +3(2)] =0 +2 +2.8 =4.8.$

$C1 =\min \{5.4, 4.8, 5.6, 4.8\} =4.8.$

Note that $C_1^{(2)}$ and $C_1^{(4)}$ tie as the minimum, giving C_1. This means that the policies corresponding to $C_1^{(2)}$ and $C_1^{(4)}$ tie as being the optimal policies. The $C_1^{(4)}$ policy says to produce enough in period 1 to cover the demand for all four periods. The $C_1^{(2)}$ policy covers only the demand through period 2. Since the latter policy has the inventory level drop to zero at the end of period 2, the C_3 result is used next, namely, produce enough in period 3 to cover the demand for periods 3 and 4. The resulting production schedules are summarized below.

Optimal Production Schedules.

1. Produce 10 airplanes in period 1.

Total variable cost _ $4.8 million.

2. Produce 5 airplanes in period 1 and 5 airplanes in period 3.

Total variable cost _ $4.8 million